1. (10 marks) Solve

$$y' = \frac{x^2+3y^2}{2xy}, \quad x > 1.$$

with the initial condition $y(2) = 6$.

Homogeneous: $z = \frac{y}{x} \Rightarrow y = zx$

$y' = z'x + z.$

$$\Rightarrow z'x + z = \frac{1+3z^2}{2z}.$$

$$\Rightarrow z'x = \frac{1+3z^2-2z^2}{2z} = \frac{1+z^2}{2z}.$$

$$\Rightarrow \frac{dz \cdot 2z}{1+z^2} = \frac{dx}{x}$$

$$\Rightarrow \ln(1+z^2) = \ln(x) + C$$

$$1+z^2 = Kx$$

$$z^2 = Kx - 1$$

$$\frac{y}{x} = z = \pm\sqrt{Kx-1}.$$

$$\Rightarrow y = \pm x\sqrt{Kx-1}.$$

$y(2)=6 \Rightarrow 6 = 2\sqrt{K2-1}$

$$36 = 4(K2-1)$$

$$9 = K \cdot 2 - 1 \Rightarrow K = 5$$

$$\boxed{y = x\sqrt{5x-1}}$$

2. (10 marks) Solve the initial value problem

$$x(2+x)y' + 2(1+x)y = 1 + 3x^2, \quad y(-1) = -1$$

$$y' + \frac{2+2x}{2x+x^2}\, y = \frac{1+3x^2}{2x+x^2}$$

Note that $(2x+x^2)' = 2+2x$.

$$\Rightarrow \left((2x+x^2)\, y\right)' = (2x+x^2)y' + (2+2x)y = 1+3x^2$$

$$\Rightarrow (2x+x^2)y = x + x^3 + c.$$

$$\Rightarrow y = \frac{x+x^3+c}{2x+x^2}$$

$$(-1) = \frac{-2+c}{-2+1} \Rightarrow 1 = -2+c$$

$$\underline{c = 3.}$$

$$\boxed{y = \frac{x+x^3+3}{2x+x^2}}$$

3. (10 marks) Find the appropriate integrating factor of the form $x^\alpha y^\beta$ for the equation

$$(3 - 20x^2y)dx + (2xy^{-1} - 12x^3)dy = 0$$

Solve the equation for the initial conditions $y(1) = 2$.

$$\underbrace{(3x^\alpha y^\beta - 20x^{\alpha+2}y^{\beta+1})}_{M}dx + \underbrace{(2x^{\alpha+1}y^{\beta-1} - 12x^{\alpha+3}y^\beta)}_{N}dy = 0$$

$$M_y = 3\beta x^\alpha y^{\beta-1} - 20(\beta+1)x^{\alpha+2}y^\beta$$
$$N_x = 2(\alpha+1)x^\alpha y^{\beta-1} - 12(\alpha+3)x^{\alpha+2}y^\beta.$$

$$M_y = N_x \iff \left.\begin{array}{l} 3\beta = 2(\alpha+1) \\ 20(\beta+1) = 12(\alpha+3). \end{array}\right\} \quad \begin{array}{l} 2\alpha - 3\beta = -2 \\ 12\alpha - 20\beta = -16 \\ \Rightarrow 3\alpha - 5\beta = -4. \\ \Rightarrow \alpha - 2\beta = -2. \end{array}$$

$$\Rightarrow \left| \begin{array}{l} \beta = 2 \\ \alpha = 2 \end{array}\right.$$

Check $(3x^2y^2 - 20x^4y^3)dx + (2x^3y - 12x^5y^2)dy.$

$$M_y - N_x = 6x^2y - 60x^4y^2 - 6x^2y + 60x^4y^2 = 0.$$

Integrate: $x^3y^2 - 4x^5y^3 = c.$

$x = 1, y = 2$: $4 - 4 \cdot 8 = c \Rightarrow c = -28.$

$$\boxed{x^3y^2 - 4x^5y^3 = -28}$$

4. (10 marks) Find the general solution to

$$y^{iv} - 6y''' + 9y'' = x$$

$y = e^{\alpha x} \Rightarrow \alpha^4 - 6\alpha^3 + 9\alpha^2 = 0$ for homog.

$\Rightarrow (\alpha - 3)^2 \alpha^2 = 0 \Rightarrow \alpha = 0$ (twice)

$\alpha = 3$ (twice).

$\Rightarrow y_1 = 1,\ y_2 = x,\ y_3 = e^{3x},\ y_4 = xe^{3x}$.

Particular sol'n for inhomogeneous:

1, x solutions $\Rightarrow$ try $y = Ax^3 + Bx^2$

$y' = 3Ax^2 + 2Bx$

$y'' = 6Ax + 2B$

$y''' = 6A$

$y^{iv} = 0$.

$\Rightarrow -36A + (54Ax + 18B) = x$.

$\Rightarrow A = \frac{1}{54} \quad B = \frac{1}{27}$.

$\Rightarrow$ gen'l Sol:

$$y = A + Bx + Ce^{3x} + Dxe^{3x} + \frac{x^3}{54} + \frac{x^2}{27}.$$

5. (10 marks) Find the general solution of

$$y'' - 2y' + y = e^x/(1+x^2)$$

Homogeneous: $\alpha^2 - 2\alpha + 1 = 0 \Rightarrow y_1 = e^x$, $y_2 = xe^x$.

$$\Rightarrow W = \begin{vmatrix} e^x & e^x \\ xe^x & (x+1)e^x \end{vmatrix} = e^{2x}.$$

Inhomogeneous: $y = u_1 e^x + u_2 x e^x$.

$$\Rightarrow u_1' = \frac{-e^x}{1+x^2} \cdot \frac{xe^x}{e^{2x}} = \frac{-x}{1+x^2} \Rightarrow u_1 = \frac{-1}{2}\ln(1+x^2)$$

$$u_2' = \frac{e^x}{1+x^2}\,\frac{e^x}{e^{2x}} = \frac{1}{1+x^2} \Rightarrow u_2 = \operatorname{atan}(x).$$

$$\Rightarrow \boxed{y = Ae^x + Bxe^x + \left(\frac{-\ln(1+x^2)}{2}\right)e^x + \operatorname{atan}(x)\,xe^x}$$

6. (10 marks) Using Laplace transforms and the following table, solve the initial value problem

$$y'' + y = u(t - \pi/2)e^t + \delta(t - \pi), \quad y(0) = y'(0) = 0$$

function $f(t)$	Laplace transform $F(s)$
1	$1/s \quad (s > 0)$
t^n	$n!/s^{n+1} \quad (s > 0)$
e^{at}	$1/(s - a) \quad (s > a)$
$\sin at$	$a/(s^2 + a^2) \quad (s > 0)$
$\cos at$	$s/(s^2 + a^2) \quad (s > 0)$
$e^{-at} f(t)$	$F(s + a)$
$u(t - a)$ or $u_a(t)$ $(a \geq 0)$	$e^{-as}/s \quad (s > 0)$
$\delta(t - a) \quad (a > 0)$	e^{-as}
$u(t - a)f(t - a)$ or $u_a(t)f(t - a)$	$e^{-as}F(s)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots - f^{(n-1)}(0)$
$f * g(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau$	$F(s)G(s)$

$$y'' + y = u(t-\tfrac{\pi}{2})e^{t-\frac{\pi}{2}} \cdot e^{\frac{\pi}{2}} + \delta(t-\pi)$$

$$\underline{\mathcal{L}}: (s^2+1)Y = e^{-\frac{\pi}{2}s} \cdot \frac{1}{s-1} \cdot e^{\frac{\pi}{2}} + e^{-\pi s}.$$

$$Y = e^{\frac{\pi}{2}} e^{-\frac{\pi}{2}s} \cdot \frac{1}{(s-1)} \frac{1}{s^2+1} + e^{-\pi s} \frac{1}{s^2+1}$$

$$= e^{\frac{\pi}{2}} e^{-\frac{\pi}{2}s} \left(\frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2+1} \right) + e^{-\pi s} \cdot \frac{1}{s^2+1}.$$

$$\Rightarrow y = e^{\frac{\pi}{2}} \left[\frac{1}{2} u(t-\tfrac{\pi}{2}) e^{(t-\frac{\pi}{2})} - \frac{1}{2} u(t-\tfrac{\pi}{2}) \cos(t-\tfrac{\pi}{2}) - \frac{1}{2} u(t-\tfrac{\pi}{2}) \sin(t-\tfrac{\pi}{2}) \right]$$

$$+ u(t-\pi) \sin(t-\pi).$$

7. (10 marks)

The 5×5 matrix

$$\begin{pmatrix} 1 & 2 & 5 & 1 & -5 \\ 1 & 2 & 6 & 3 & -7 \\ -1 & -2 & -5 & 0 & 6 \\ 3 & 6 & 14 & 2 & -12 \\ 1 & 2 & 1 & -7 & 3 \end{pmatrix}$$

row reduces to

$$\begin{pmatrix} 1 & 2 & 5 & 1 & -5 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 5 & 0 & -6 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 & 14 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Compute bases of the image and the kernel of M.

Image. take columns of original matrix corresponding to pivot columns in reduced matrix:

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ -5 \\ 14 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \\ -7 \end{pmatrix}.$$

Kernel : Take reduced echelon form, and use free variables:

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -14 \\ 0 \\ 4 \\ -1 \\ 1 \end{pmatrix},$$

8. (10 marks) Apply the Gram-Schmidt method to compute an orthonormal basis of the vector space spanned by

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ 1 \\ -5 \\ 3 \end{pmatrix}$$

① Orthogonal basis:

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{\langle v_1, v_2 \rangle}{\langle v_1, v_1 \rangle} v_1 = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix} - \frac{-6}{6} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$u_3 = v_3 - \frac{\langle v_1, v_3 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} . u_2$$

$$= \begin{pmatrix} -1 \\ 1 \\ -5 \\ 3 \end{pmatrix} - \frac{-12}{6} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} - \frac{6}{6} \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

② Normalise

$$e_1 = \frac{v_1}{\langle v_1, v_1 \rangle^{1/2}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$e_2 = \frac{u_2}{\langle u_2, u_2 \rangle^{1/2}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$e_3 = \frac{u_3}{\langle u_3, u_3 \rangle^{1/2}} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \end{pmatrix}.$$

9. (10 marks) Find the eigenvalues and corresponding eigenvectors of the symmetric matrix

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

Find an orthogonal matrix P and a diagonal matrix D with $A = PDP^T$.

Compute the matrices in the standard basis of orthogonal projection onto each eigenspace.

<u>Eigenvalues</u> $\det\begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = (4-\lambda)^2 - 4 = 12 - 8\lambda + \lambda^2 = 0$

$$\Rightarrow \lambda = \frac{8 \pm \sqrt{64-48}}{2} = 6, 2.$$

<u>Eigenvectors</u> <u>$\lambda = 6$:</u> $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \Rightarrow \quad$ take $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

<u>$\lambda = 2$.</u> $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \Rightarrow \quad$ take $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

<u>then</u> $D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

<u>Projections</u> : $\Pi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$= v_1 \cdot v_1^T = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

$$\Pi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= v_2\, v_2^T = \frac{1}{2} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

10. (10 marks)
Find the general solution to the linear system

$$Y' = \begin{pmatrix} 4 & -5 & 0 \\ 5 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix} \cdot Y$$

Express your basis of solutions in real form (i.e no complex exponentials). Which solutions decay to 0 as x goes to $+\infty$? Solve for the initial conditions $Y(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Eigenvalues : $\det\begin{pmatrix} 4-\lambda & -5 & 0 \\ 5 & -4-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{pmatrix} = \left(\lambda^2 - 16 + 25\right)(-2-\lambda)$.

$\Rightarrow \lambda = \pm 3i, \ -2$.

$\lambda = -2$: $\begin{pmatrix} 6 & -5 & 0 \\ 5 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \Rightarrow \quad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 3i$: $\begin{pmatrix} 4-3i & -5 & 0 \\ 5 & -4-3i & 0 \\ 0 & 0 & -2-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \Rightarrow v = \begin{pmatrix} 5 \\ 4-3i \\ 0 \end{pmatrix}$

$\lambda = -3i$ Take c-conjugate $\Rightarrow v = \begin{pmatrix} 5 \\ 4+3i \\ 0 \end{pmatrix}$

$\Rightarrow$ Solution $Y_1 = \begin{pmatrix} 0 \\ 0 \\ e^{-2t} \end{pmatrix}$

$$Y_2 = \begin{pmatrix} 5\left(\dfrac{e^{3it} + e^{-3it}}{2}\right) \\ +4\left(\dfrac{e^{3it} + e^{-3it}}{2}\right) + 3i^2\left(\dfrac{e^{3it} - e^{3it}}{2i}\right) \\ 0 \end{pmatrix} = \begin{pmatrix} 5\cos(3t) \\ 4\cos(3t) + 3\sin(3t) \\ 0 \end{pmatrix}$$

$$Y_3 = \begin{pmatrix} 5\left(\dfrac{e^{3it} - e^{-3it}}{2i}\right) \\ 4\left(\dfrac{e^{3it} - e^{3it}}{2i}\right) - 3\cancel{i}\left(\dfrac{e^{3it} + e^{3it}}{2\cancel{i}}\right) \\ 0 \end{pmatrix} = \begin{pmatrix} 5\sin(3t) \\ 4\sin(3t) - 3\cos(3t) \\ 0 \end{pmatrix}$$

$\rightarrow Y(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow Y = \frac{-1}{3} Y_3$

$\rightarrow Y(x) \rightarrow 0$ as $x \rightarrow \infty$: $Y = c\, Y_1$